GEOMETRIC PROOF OF WIGNER'S THEOREM

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Abstract. Great stuff.

1. Introduction

Projective geometry has proven itself to be an extremely useful tool and has been at the root of many beautiful results in both mathematics and physics. Here we will work through one of the smoothest applictations of projective geometry in a proof of Wigner's theorem, which states that any quantum mechanical sysmmetry is represented by either a unitary or anti-unitary operator. For now, we informally define a quantum symmetry as any transformation acting on the pure states of a quantum system which preserves the transiation probability. We will construct a mathematically rigorous definition from this loose physical one later in Section 2. Presently, we give a brief overview of projective geometry and the specific notation required for the proof.

Consider the vector space V of dimension $n+1$ over the set of complex numbers $\mathbb{C}$. We define the projective space $\mathbb{P}V$ corresponding to V as the space of *rays*, L_i, representing the equivalences classes $[x]$ of the relation $x' = cx$. Specifically, the projection[1]

$$\pi : V \to \mathbb{P}\,V \qquad x \mapsto [x] \tag{1.1}$$

takes us between a vector space and its projective space. Two position vectors $x = (x_1, x_2, \ldots, x_n)$ and $x' = (x'_1, x'_2, \ldots, x'_n)$ in V represent the same ray in $\mathbb{P}V$ if and only if $x'_i = c_i x_i,\ \forall i$ (reference: Shripad Thite). In this sense, each ray in projective space corresponds to a one-dimensional subspace of the vector space. Thus, we see that the projective space $\mathbb{P}V$ is of dimension n. [2]

Definition 1.1. A set of rays $\{L_i\}$ in projective space $\mathbb{P}V$ is said to be *projectively independent* if and only if there exists a linearly independent set

[1]Recall that a projection of this type will, in fact, be surjective and continuous.

[2]Up until this point, it has been most natural to think of the obejects representing these equivlance classes as lines. However, later, it will be better to pick representatives from these lines and, instead of talking about the rays in the projective space, we will talk about the points in the projective space. In this sense, we think of all objects in the projective space as having one dimension less than their corresponding object in the vector space that we are projecting from.

of vectors $\{a_i\}$ such that for $k = 1, 2, \ldots, m$, we have $a_k \in L_k$. In this case, the vectors $\{a_i\}$ span an m-dimensional subspace of the vector space V.

Remark 1.2. If the set of vectors $\{a_i\}$ is not linearly independent, then one of them can be written as a linear combination of the others, which means that it is in an equivalence class which defines *both* of the rays as defined above. This tells us that two rays in projective space are either projectively independent or they represent the same rays.

Definition 1.3. In the projective space $\mathbb{P}V$ of dimension n, a set $B := \{b_1, b_2, \ldots, b_{n+2}\} \subseteq \mathbb{P}V$ of $n+2$ rays in the projective space is called a *base* of the projective space if and only if any subset of B containing $n+1$ rays is projectively independent.

Next, we define a useful operation, called the *unification* of two projectively independent rays L_1 and L_2, which defines the subspace spanned by L_1 and L_2 and is denoted

$$L_1 \vee L_2 := \{[\ell_1 + \ell_2] : \ell_1 \in L_1, \ell_2 \in L_2\} \tag{1.2}$$

We will refer to the unification of L_1 and L_2 as the *projective line* and we say that this projective line is uniquely determined by the two *projective points* L_1 and L_2. Note that three projective points are said to be *collinear* if they all fall on the same projective line.

Definition 1.4. A *collineation* is a bijective map $\Upsilon : \mathbb{P}V \to \mathbb{P}W$ between two projective spaces which preserves collinearity. In other words, a collineation maps projective lines in $\mathbb{P}V$ to projective lines in $\mathbb{P}W$:

$$\Upsilon(L_1 \vee L_2) = \Upsilon(L_1) \vee \Upsilon(L_2) \tag{1.3}$$

Definition 1.5. A *semi-projectivity* is a bijective map $\phi : \mathbb{P}V \to \mathbb{P}W$ which is induced by a semi-linear map $\Phi : V \to W$. In other words,

$$[\Phi \cdot \ell_1] = \phi[\ell_1] \tag{1.4}$$

In this case, Φ is said to be *compatible* with ϕ.

The prefix "semi" refers to the fact that the map Φ is a linear map *up to automorphism*. In other words, for τ some automorphism of $\mathbb{C}$, $\lambda_1, \lambda_2 \in \mathbb{C}$, and $v_1, v_2 \in V$, we have that

$$\Phi(\lambda_1 v_1 + \lambda_2 v_2) = \tau(\lambda_1)v_1 + \tau(\lambda_2)v_2 \tag{1.5}$$

Now, in the world of quantum mechanics, our V is actually a Hilbert space $\mathcal{H}$. We will show in the coming sections that the only automorphisms of $\mathcal{H}$ are the identity map and the complex conjugation map. So, in the quantum world, the fact that our compatible map Φ is semi-linear actually means that it is either linear or anti-linear. We will comment more on this in Section 4.

Now, due to the semi-linear nature of Φ, it is easy to see that every semi-projectivity is a collineation. However, is the converse statement true? The answer to this question is called the Fundamental Theorem of Projective

Geometry, and will be the core of the proof (and the main source of beauty) in Section 4.

Theorem 1.6. *Any collineation* $\Upsilon : \mathbb{P}V \to \mathbb{P}W$*, where* $\mathbb{P}V$ *and* $\mathbb{P}W$ *are finite-dimensional projective spaces of dimension* $n \geq 2$*, is a semi-projectivity. Pictorally, this is the statement that the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{\Upsilon} & W \\ \big\downarrow & & \big\downarrow \\ \mathbb{P}V & \xrightarrow[\Phi]{} & \mathbb{P}W \end{array}$$

Comment: Make a short comment on the various proofs of this theorem, dimensionality, where to find them, general idea, etc.

2. Quantum Mechanics meets Projective Geometry

Comment: Develop the projection of the pure states in Hilbert space onto the unit sphere using group-theoretic notions, mobis transformations, etc.

The pure states can be described as one-dimensional subspaces of the corresponding Hilbert space. In other words, the rays of the projective Hilbert space $\mathbb{P}\mathcal{H}$ endowed with the inner product $\langle \cdot | \cdot \rangle$ represent the space of pure states of a quantum mechanical system. In $\mathbb{P}\mathcal{H}$, we have a function $p : \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \to [0,1]$ which computes the *transition probability* of two normalized quantum states in our projective Hilbert space. In other words, if ψ_i represents the unit vector in the direction of L_i in $\mathbb{P}\mathcal{H}$, then

$$p(L_1, L_2) = |\langle \psi_1 | \psi_2 \rangle|^2 \tag{2.1}$$

We have now come to the point where we can make rigorous the notion of a quantum mechanical symmetry.

Definition 2.1. A *symmetry transformation* (*i.e.* quantum symmetry) is a bijective map $T : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}'$ such that

$$p(T \cdot L_1, T \cdot L_2) = |{}_T\langle \psi_1 | \psi_2 \rangle_T|^2 = |\langle \psi_1 | \psi_2 \rangle|^2 = p(L_1, L_2), \tag{2.2}$$

where $|\psi_i\rangle_T$ represents the state in $\mathcal{H}'$ which is the transformation of $|\psi_i\rangle$ under T.

3. A Restatement of Wigner's Theorem

Now comes the time to formally state the theorem that we would like to prove. Informally, we have stated Wigner's theorem by saying that any quantum symmetry is represented by a unitary or antiunitary operator. Specifically, we're saying that any quantum symmetry T acting on the pure states in projective space lifts to either a unitary or anti-unitary operator

U, which acts on our pure states in the standard Hilbert space. We will now restate this in terms similar to the Theorem 1.6:

Theorem 3.1 (Wigner)**.** *Any quantum symmetry $T : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}'$ has a compatible transformation $U : \mathcal{H} \to \mathcal{H}'$ which is either unitary or anti-unitary. Pictorally, this says that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{U} & \mathcal{H}' \\ \big\downarrow & & \big\downarrow \\ \mathbb{P}\mathcal{H} & \xrightarrow[T]{} & \mathbb{P}\mathcal{H}' \end{array}$$

4. The Proof of Wigner's Theorem

Lemma 4.1. *Any quantum symmetry is a collineation.*

Proof. I have the proof written down and will type it in soon. □

Lemma 4.2. *Any semi-linear transformation between two hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ is either unitary or anti-unitary.*

Proof. See above. □

We will now see that Wigner's theorem falls out almost as a corollary of the Fundamental Theorem of Projective Geometry.

Proof of Theorem 3.1. Suppose we have some quantum symmetry $T : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}'$. By Lemma 4.1, T is a collineation and so, by the Fundamental Theorem of Projective Geometry, T is also a semi-projectivity. This means that T is induced by a semi-linear transformation $U : \mathcal{H} \to \mathcal{H}'$, which, by Lemma 4.2, must be either unitary or anti-unitary. This concludes the proof of Wigner's theorem for a finite dinemsional Hilbert space.

For the infinite-deminsional case, we need to work out a couple of formalities. Note that all of the previous steps hold *except* for Theorem 1.6. The theorem breaks down in using U as a compatible semi-linear map to T. So we need to show that given some semi-linear[3] operator $U : \mathcal{H} \to \mathcal{H}'$ is compatible with the quantum symmetry $T : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}'$, where $\mathcal{H}$ and $\mathcal{H}'$ as well as their corresponding projective spaces are now infinite-dimensional. In other words, we need to show that

$$T[a] = [Ua] \quad \forall\ a \in \mathcal{H} \tag{4.1}$$

[3]We will invoke Lemma4.2 again here, since it holds for infinite dimensional cases as well.

To show this, let v_i and v_i' be countable bases for $\mathcal{H}$ and $\mathcal{H}'$ respectively. Then, from the finite-deminsional case, we know that

$$T\left[\sum_{i=1}^{n} \lambda_i v_i\right] = \left[\sum_{i=1}^{n} \lambda_i U v_i\right] \tag{4.2}$$

Furthermore, since the projection map taking the lines $\ell \in \mathcal{H}$ to the projective points $[\ell] \in \mathbb{P}\mathcal{H}$ is, by definition, continuous, we know that for all sequences $(x_n)_{n\in\mathbb{N}} \in \mathcal{H}$,

$$\left[\lim_{n\to\infty} (x_n)\right] = \lim_{n\to\infty} [x_n] \tag{4.3}$$

Thus, we can finally write

(4.4)

$$T[a] = T\left[\lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i v_i\right] = \lim_{n\to\infty} T\left[\sum_{i=1}^{n} \lambda_i v_i\right] = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \lambda_i U v_i\right] = [Ua]$$

Since compatibility is the only point where the finite dimensional proof fell through when generalizing to the infinite-dimensional case, this completes the proof. □

5. Conclusion

References

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